

SCATTERING CROSS-SECTIONS IN QUANTUM GRAVITY - THE CASE OF MATTER-MATTER SCATTERING

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Abstract

Viewing gravitational energy-momentum p_G^μ as equal by observation, but different in essence from inertial energy-momentum p_I^μ naturally leads to the gauge theory of volume-preserving diffeomorphisms of an inner Minkowski space. To analyse scattering in this theory the gauge field is coupled to two Dirac fields with different masses. Based on a generalized LSZ reduction formula the S -matrix element for scattering of two Dirac particles in the gravitational limit and the corresponding scattering cross-section are calculated to leading order in perturbation theory. Taking the non-relativistic limit for one of the initial particles in the rest frame of the other the Rutherford-like cross-section of a non-relativistic particle scattering off an infinitely heavy scatterer calculated quantum mechanically in Newtonian gravity is recovered. This is a further indication that the gauge field theory of volume-preserving diffeomorphisms can be viewed as a quantum theory of gravity.

1 Introduction

In two papers we have developed the gauge field theory of volume-preserving diffeomorphisms of an inner Minkowski space both at the classical [1, 2] and the quantum level [3, 4] treating inertial energy-momentum p_I^μ and gravitational energy-momentum p_G^μ as two separate observables which happen to be equal by measurement, but play fundamentally different roles in the theory - p_I^μ describing the state of motion of a particle both in the absence and presence of interactions and p_G^μ providing both the source for gravitational interaction and experiencing it. We In [1, 3] we have shown that it is possible to deal with the challenges posed by a non-compact gauge group such as dealing with the infinite group volume or ensuring positive gauge field energy. On the other hand the well-developed apparatus of both gauge [5, 6] and quantum field theory [7, 8, 9] can be relatively easily generalized to the case of the gauge theory of volume-preserving diffeomorphisms and on that basis we have the theory shown to be renormalizable. At one loop the pure gauge field theory is asymptotic free whereas adding the Standard Model content of matter and other gauge

fields destroys the asymptotic freedom of the combined theory [3], i.e. the gauge field quanta are not confined as we would expect for a viable theory of gravity.

In a further paper [10] we have constructed the asymptotic states belonging to both matter fields and the gauge field. The one-particle matter states carry besides their usual conserved quantum numbers such as energy-momentum, spin and inner Noether charges an additional conserved four-vector we call inner momentum - which is to be identified with gravitational energy-momentum. The one-particle gauge field states carry energy-momentum, spin and inner momentum as well as inner spin. To account for the observed equality of inertial and gravitational energy-momentum we have defined the so-called gravitational limit which identifies energy-momentum and inner momentum for the asymptotic states. In this limit these states become observable with inertial and gravitational energy-momentum equal as dictated by experiment and the gauge field becomes a spin-2 fields. Finally we have defined the S -matrix relating these observable states and derived generalized LSZ reduction formulae relating physical scattering amplitudes to the gravitational limit of truncated on shell Fourier-transformed vacuum expectation values of time-ordered products of field operators in the interacting theory.

Why, however, and on what basis do we claim this to be a potential quantum theory of gravity?

On one hand it fulfils general requirements such as the universal coupling to all other fields [1, 3]. On the other if it is a theory of gravity we should be able to analyse physical situations for which we can make and compare predictions both within the framework of the theory presented as well as within the standard framework of Newtonian gravity dealt with quantum-mechanically. As we can evaluate the gauge theory of volume-preserving diffeomorphisms only perturbatively and only for the scattering of particles at this point one situation of physical interest is the gravitational scattering of two particles with different masses. To be specific we should be able to calculate the scattering cross-section of two Dirac particles with different masses and to reproduce in an appropriate limit the cross-section of a non-relativistic particle scattering off an infinitely heavy scatterer calculated quantum mechanically in Newtonian gravity - which is the content of this paper.

2 Matter-Matter Scattering Amplitude

In this section we calculate the scattering amplitude of two Dirac particles with different masses in quantum gravity to lowest order in perturbation

theory in natural units $c = \hbar = 1$.

Our starting point is the action for two Dirac fields ψ and Ψ with masses m and M respectively which are coupled to the gravitational field $A^\mu{}_\alpha$ as we have generally defined it in [1, 3, 10]

$$\begin{aligned}
S = & \int d^4x \int d^4X \Lambda^{-4} \left\{ \frac{1}{4} F_{\mu\nu}{}^\alpha(x, X) \cdot F^{\mu\nu}{}_\alpha(x, X) \right. \\
& + \frac{\lambda}{2} \partial_\mu A^\mu{}_\alpha(x, X) \cdot \partial^\nu A_\nu{}^\alpha(x, X) \\
& - \frac{\mu^2}{2} A^\mu{}_\alpha(x, X) \cdot A_\mu{}^\alpha(x, X) \\
& + \bar{\psi}(x, X) \left(\frac{i}{2} \not{D} - \frac{i}{2} \not{D} - m \right) \psi(x, X) \\
& \left. + \bar{\Psi}(x, X) \left(\frac{i}{2} \not{D} - \frac{i}{2} \not{D} - M \right) \Psi(x, X) \right\}.
\end{aligned} \tag{1}$$

Above, x and X denote spacetime and inner space coordinates [1, 3, 10],

$$\begin{aligned}
F_{\mu\nu}{}^\alpha(x, X) = & \partial_\mu A_\nu{}^\alpha(x, X) - \partial_\nu A_\mu{}^\alpha(x, X) \\
& + g\Lambda A_\mu{}^\beta(x, X) \cdot \nabla_\beta A_\nu{}^\alpha(x, X) - g\Lambda A_\nu{}^\beta(x, X) \cdot \nabla_\beta A_\mu{}^\alpha(x, X)
\end{aligned} \tag{2}$$

denotes the gravitational field strength and

$$D_\mu = \partial_\mu + g\Lambda A_\mu{}^\alpha \cdot \nabla_\alpha \tag{3}$$

the covariant derivative [1, 3, 10], g a dimensionless coupling constant and Λ a length scale in inner space [1]. Note that we have added both a gauge-fixing term proportional to a constant λ and a mass term for the gauge field with mass μ to deal with possible infrared problems [10]. All other notations and conventions have been collected in Appendix A.

We want to calculate the scattering amplitude of two Dirac particles with incoming and outgoing inertial equal to gravitational energy-momenta p_i, q_i , and p_f, q_f respectively, incoming and outgoing spins γ_i, γ'_i and γ_f, γ'_f respectively and masses m ($p_i^2 = p_f^2 = m^2$) and M ($q_i^2 = q_f^2 = M^2$). Above i and f refer to initial and final states.

In quantum gravity S -matrix elements are related by generalized LSZ reduction formulae [10] to the gravitational limit of truncated on shell Fourier-transformed vacuum expectation values of time-ordered products of field operators in the interacting theory. Applying the general expression Eqn.(149) for generalized Dirac matter LSZ reduction formulae in [10] to the case at hands the amplitude is found to be

$$\begin{aligned}
\langle p_f, q_f \text{ out} | p_i, q_i \text{ in} \rangle &= \lim_{\mu \rightarrow 0} \lim_{P_f \rightarrow p_f} \lim_{P_i \rightarrow p_i} \lim_{Q_f \rightarrow q_f} \lim_{Q_i \rightarrow q_i} \\
&\left(\frac{i}{\sqrt{Z_2}} \right)^2 \int d^4 x_i \int d^4 X_i \Lambda^{-4} \int d^4 x_f \int d^4 X_f \Lambda^{-4} \\
&\left(\frac{i}{\sqrt{Z_2}} \right)^2 \int d^4 y_i \int d^4 Y_i \Lambda^{-4} \int d^4 y_f \int d^4 Y_f \Lambda^{-4} \\
&\bar{u}(p_f, \gamma_f) e^{ip_f x_f + i P_f X_f} \left(i \vec{\partial}_{x_f} - m \right) \\
&\bar{U}(q_f, \gamma'_f) e^{iq_f y_f + i Q_f Y_f} \left(i \vec{\partial}_{y_f} - M \right) \\
\langle 0 | T \left(\bar{\psi}(x_i, X_i) \psi(x_f, X_f) \bar{\Psi}(y_i, Y_i) \Psi(y_f, Y_f) \right) | 0 \rangle & \\
&\left(-i \vec{\partial}_{x_i} - m \right) u(p_i, \gamma_i) e^{-ip_i x_i - i P_i X_i} \\
&\left(-i \vec{\partial}_{y_i} - M \right) U(q_i, \gamma'_i) e^{-iq_i y_i - i Q_i Y_i}.
\end{aligned} \tag{4}$$

Above $u(p, \gamma)$ and $U(q, \gamma')$ denote free Dirac spinors describing the asymptotic states of the fields ψ and Ψ with momenta p, q and spins γ, γ' respectively and Z_2 the spinor field renormalization constant.

Next we have to calculate the time-ordered product of the four interacting field operators in Eqn.(4) which is obtained from the generating functional $\mathcal{Z}[\eta, \bar{\eta}; H, \bar{H}; J]$ for the Green functions in quantum gravity by

$$\begin{aligned}
\langle 0 | T \left(\bar{\psi}(x_i, X_i) \psi(x_f, X_f) \bar{\Psi}(y_i, Y_i) \Psi(y_f, Y_f) \right) | 0 \rangle &= \\
\frac{\Lambda^4 \delta}{i \delta \bar{\eta}(x_f, X_f)} \frac{\Lambda^4 \delta}{i \delta \bar{H}(y_f, Y_f)} \mathcal{Z}[\eta, \bar{\eta}; H, \bar{H}; J] \frac{\Lambda^4 \delta}{i \delta \eta(x_i, X_i)} \frac{\Lambda^4 \delta}{i \delta H(y_i, Y_i)},
\end{aligned} \tag{5}$$

where η, H, J denote external sources for the fields ψ, Ψ, A to which they are coupled through linear terms in the action. \mathcal{Z} has been defined in [3] in the path integral representation Eqn.(46) in that paper. We now turn to evaluate it perturbatively in the usual way

$$\begin{aligned}
\mathcal{Z}[\eta, \bar{\eta}; H, \bar{H}; J] &= \exp i S_{INT} \left[\frac{\Lambda^4 \delta}{i \delta \eta}, \frac{\Lambda^4 \delta}{i \delta \bar{\eta}}, \frac{\Lambda^4 \delta}{i \delta H}, \frac{\Lambda^4 \delta}{i \delta \bar{H}}, \frac{\Lambda^4 \delta}{i \delta J} \right] \\
&\mathcal{Z}_0[\eta, \bar{\eta}; H, \bar{H}; J].
\end{aligned} \tag{6}$$

Above \mathcal{Z}_0 is the generating functional for free Green functions as given by Eqn.(50) in [3] and S_{INT} the interaction part of the action S in Eqn.(1) cubic and quartic in the fields. \mathcal{Z}_0 is easily calculated to be [3]

$$\begin{aligned}
\mathcal{Z}_0[\eta, \bar{\eta}; H, \bar{H}; J] &\propto \exp \frac{i}{2} \iint J_\mu^\alpha \cdot G_F^{\mu\nu}{}_{\alpha\beta} J_\nu^\beta \\
&\exp i \iint \bar{\eta} \cdot S_F^m \eta \cdot \exp i \iint \bar{H} \cdot S_F^M H,
\end{aligned} \tag{7}$$

where S_F^m and $G_F^{\mu\nu}{}_{\alpha\beta}$ denote the free Dirac propagator as in Eqn.(65) in [10]

$$S_F^m(x, X) = i \Lambda^4 \delta^4(X) \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \frac{\not{k} + m}{k^2 - m^2 + i\varepsilon} \quad (8)$$

and gauge field propagator as in Eqn.(102) in [10] for the choice of gauge made in that paper

$$G_F^{\mu\nu}{}_{\alpha\beta}(x, X) = -i {}^T\delta_{\alpha\beta}(X) \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \frac{\eta^{\mu\nu}}{k^2 - \mu^2 + i\varepsilon} \quad (9)$$

for the fields ψ and $A_\mu{}^\alpha$ respectively with an expression analogous to Eqn.(8) for the propagator of the field Ψ . Above ${}^T\delta_{\alpha\beta}(X)$ refers to the delta function transversal in inner space

$${}^T\delta_{\alpha\beta}(X) = \int \frac{d^4 K}{(2\pi)^4} \Lambda^4 e^{-iKX} \left(\eta_{\alpha\beta} - \frac{K_\alpha K_\beta}{K^2} \right) \quad (10)$$

introduced in [10].

The part of S_{INT} relevant to our calculation is

$$\begin{aligned} S_{INT} = & \int d^4 x \int d^4 X \Lambda^{-4} \frac{ig\Lambda}{2} \left\{ \text{other terms} \right. \\ & + \bar{\psi}(x, X) \left(\gamma^\mu A_\mu{}^\alpha(x, X) \vec{\nabla}_\alpha - \gamma^\mu \overleftarrow{\nabla}_\alpha A_\mu{}^\alpha(x, X) \right) \psi(x, X) \\ & \left. + \bar{\Psi}(x, X) \left(\gamma^\mu A_\mu{}^\alpha(x, X) \vec{\nabla}_\alpha - \gamma^\mu \overleftarrow{\nabla}_\alpha A_\mu{}^\alpha(x, X) \right) \Psi(x, X) \right\}. \end{aligned} \quad (11)$$

Note the arrows on the derivatives w.r.t. inner coordinates indicating the directions in which they act.

Evaluating the functional derivatives in Eqns.(5) and (6), setting the source terms equal to zero and discarding disconnected and higher order contributions we obtain the vacuum expectation value for the time-ordered product of the four field operators to leading order

$$\begin{aligned} \langle 0 | T \left(\bar{\psi}(x_i, X_i) \psi(x_f, X_f) \bar{\Psi}(y_i, Y_i) \Psi(y_f, Y_f) \right) | 0 \rangle = & \\ & - \left(\frac{ig\Lambda}{2} \right)^2 \int d^4 x \int d^4 X \Lambda^{-4} \int d^4 y \int d^4 Y \Lambda^{-4} \\ & \left[S_F^m(x_f - x, X_f - X) \gamma_\mu \left(\vec{\nabla}_x^\alpha S_F^m(x - x_i, X - X_i) \right) \right. \\ & - \left(S_F^m(x_f - x, X_f - X) \gamma_\mu \overleftarrow{\nabla}_x^\alpha \right) S_F^m(x - x_i, X - X_i) \Big] \\ & i G_F^{\mu\nu}{}_{\alpha\beta}(x - y, X - Y) \\ & \left[S_F^M(y_f - y, Y_f - Y) \gamma_\nu \left(\vec{\nabla}_y^\beta S_F^M(y - y_i, Y - Y_i) \right) \right. \\ & - \left(S_F^M(y_f - y, Y_f - Y) \gamma_\nu \overleftarrow{\nabla}_y^\beta \right) S_F^M(y - y_i, Y - Y_i) \Big]. \end{aligned} \quad (12)$$

Inserting this expression in Eqn.(4) and performing the truncation we find the scattering amplitude to be

$$\begin{aligned}
\langle p_f, q_f \text{ out} | p_i, q_i \text{ in} \rangle &= \lim_{\mu \rightarrow 0} \lim_{P_f \rightarrow p_f} \lim_{P_i \rightarrow p_i} \lim_{Q_f \rightarrow q_f} \lim_{Q_i \rightarrow q_i} i(g\Lambda)^2 \\
&\int d^4x \int d^4X \Lambda^{-4} \int d^4y \int d^4Y \Lambda^{-4} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4K}{(2\pi)^4} \Lambda^4 \\
&\bar{u}(p_f, \gamma_f) \frac{i(P_i^\alpha + P_f^\alpha)}{2} \gamma_\mu u(p_i, \gamma_i) \\
&\frac{\eta^{\mu\nu}}{k^2 - \mu^2 + i\varepsilon} \left(\eta_{\alpha\beta} - \frac{K_\alpha K_\beta}{K^2} \right) \\
&\bar{U}(q_f, \gamma'_f) \frac{i(Q_i^\beta + Q_f^\beta)}{2} \gamma_\nu U(q_i, \gamma'_i) \\
&e^{ix(p_f - p_i - k)} e^{iy(q_f - q_i + k)} e^{iX(P_f - P_i - K)} e^{iY(Q_f - Q_i + K)}.
\end{aligned} \tag{13}$$

Performing the remaining integrations the amplitude finally becomes

$$\begin{aligned}
\langle p_f, q_f \text{ out} | p_i, q_i \text{ in} \rangle &= \lim_{\mu \rightarrow 0} \lim_{P_f \rightarrow p_f} \lim_{P_i \rightarrow p_i} \lim_{Q_f \rightarrow q_f} \lim_{Q_i \rightarrow q_i} (-i) (g\Lambda)^2 \\
&(2\pi)^4 \delta^4(p_f - p_i + q_f - q_i) (2\pi)^4 \Lambda^{-4} \delta^4(P_f - P_i + Q_f - Q_i) \\
&\bar{u}(p_f, \gamma_f) \gamma^\mu u(p_i, \gamma_i) \frac{1}{(p_f - p_i)^2 - \mu^2 + i\varepsilon} \bar{U}(q_f, \gamma'_f) \gamma_\mu U(q_i, \gamma'_i) \\
&\frac{P_{i\alpha} + P_{f\alpha}}{2} \left(\eta^{\alpha\beta} - \frac{(P_f^\alpha - P_i^\alpha)(Q_i^\beta - Q_f^\beta)}{(P_f - P_i)(Q_i - Q_f)} \right) \frac{Q_{i\beta} + Q_{f\beta}}{2}.
\end{aligned} \tag{14}$$

Note that before taking the gravitational limit the amplitude is scale-invariant under $P \rightarrow \rho P$, $Q \rightarrow \rho Q$ and $\Lambda \rightarrow \rho^{-1} \Lambda$ as it has to be [1, 3].

Trying to take the limits above we are left with an expression of the type $(2\pi)^4 \Lambda^{-4} \delta^4(0)$ which we also have encountered in defining asymptotic states in quantum gravity [10]. Noting that

$$(2\pi)^4 \Lambda^{-4} \delta^4(0) \sim \Lambda^{-4} \int d^4X \rightarrow \Lambda^{-4} \mathbb{V}_g \tag{15}$$

with \mathbb{V}_g being the regularized inner Minkowski space volume we use the fact that Λ is an a priori unspecified parameter which we can freely choose so that

$$\Lambda^{-4} \mathbb{V}_g = 1. \tag{16}$$

This is the regularization we employed in [10] to deal with expressions of the sort of $(2\pi)^4 \Lambda^{-4} \delta^4(0)$ and is the same as used in Fermi's trick to evaluate squares of Dirac's delta distribution when squaring amplitudes.

Noting that in the limit above

$$\begin{aligned}
&(P_{i\alpha} + P_{f\alpha})(P_f^\alpha - P_i^\alpha)(Q_i^\beta - Q_f^\beta)(Q_{i\beta} + Q_{f\beta}) = \\
&(P_f^2 - P_i^2)(Q_i^2 - Q_f^2) \rightarrow (m^2 - m^2)(M^2 - M^2) = 0
\end{aligned} \tag{17}$$

vanishes we see that the inner longitudinal part of the gauge field propagator Eqn.(10) does not contribute to the amplitude.

As there is no infrared problem for $\mu \rightarrow 0$ we can now safely take all limits. Before doing so we also invoke the inner scale invariance of the amplitude to rescale $\Lambda \rightarrow L_P$, where $L_P = \sqrt{\Gamma}$ is the Planck length in natural units $c = \hbar = 1$ and get the final expression for the scattering amplitude

$$\langle p_f, q_f \text{ out} | p_i, q_i \text{ in} \rangle = i (2\pi)^4 \delta^4(p_f - p_i + q_f - q_i) \mathcal{M}_{fi} \quad (18)$$

with the invariant matrix element \mathcal{M}_{fi} found to be

$$\mathcal{M}_{fi} = - (gL_P)^2 \bar{u}_f \gamma^\mu u_i \frac{(p_i + p_f) \cdot (q_i + q_f)}{4((p_f - p_i)^2 + i\varepsilon)} \bar{U}_f \gamma_\mu U_i. \quad (19)$$

It contains the information about the underlying dynamics of the theory and is completely symmetric under the interchange of the two particles, i.e. $p \leftrightarrow q$ and $u \leftrightarrow U$.

We note the similarity of \mathcal{M}_{fi} with the invariant matrix element for scattering of two Dirac particles with different masses in quantum electrodynamics [7, 11]. However, there is a crucial difference: the strength of the scattering in the case of quantum electrodynamics is proportional to the product of the two electric charges $e e'$ whereas in quantum gravity it is proportional to the Minkowski product of the momentum four-vectors $(gL_P)^2 \frac{(p_i + p_f) \cdot (q_i + q_f)}{4}$ which changes the dynamics completely. Note that in the rest frame of the particle with mass M the coupling strenght reduces to $(gL_P)^2 \frac{(p_i + p_f) \cdot (q_i + q_f)}{4} = (gL_P)^2 \frac{M(E + E')}{2} > (gL_P)^2 M m$.

3 Matter-Matter Scattering Cross-Section

In this section we calculate the cross-section for the scattering of two Dirac particles with different masses in quantum gravity to lowest order in perturbation theory.

We start with the usual Lorentz-invariant expression for the cross-section with two incoming and two outgoing Dirac fermions [7, 11]

$$d\sigma = \frac{m M}{\sqrt{(p_i \cdot q_i)^2 - m^2 M^2}} \frac{m}{E_{p_f}} \frac{d^3 p_f}{(2\pi)^3} \frac{M}{E_{q_f}} \frac{d^3 q_f}{(2\pi)^3} (2\pi)^4 \delta^4(p_f - p_i + q_f - q_i) |\mathcal{M}_{fi}|^2. \quad (20)$$

As we are interested in the unpolarized cross-section we first average over initial and sum over final states

$$|\mathcal{M}_{fi}|^2 \rightarrow \overline{|\mathcal{M}_{fi}|^2} \equiv \frac{1}{4} \sum_{\gamma_i, \gamma_f; \gamma'_i, \gamma'_f} |\mathcal{M}_{fi}|^2. \quad (21)$$

Proceeding with the calculation of $\overline{|\mathcal{M}_{fi}|^2}$ we encounter two expressions of the type

$$\begin{aligned} \sum_{\gamma_i, \gamma_f} \bar{u}_f \gamma^\mu u_i \bar{u}_i \gamma^\nu u_f &= \text{tr} \left(\frac{\not{p}_i + m}{2m} \gamma^\mu \frac{\not{p}_f + m}{2m} \gamma^\nu \right) \\ &= \frac{1}{m^2} (p_i^\mu p_f^\nu + p_i^\nu p_f^\mu - p_i \cdot p_f \eta^{\mu\nu} + m^2 \eta^{\mu\nu}). \end{aligned} \quad (22)$$

Inserting these and performing the Lorentz sums in $\overline{|\mathcal{M}_{fi}|^2}$ leaves us with

$$\begin{aligned} \overline{|\mathcal{M}_{fi}|^2} &= (gL_P)^4 \frac{((p_i + p_f) \cdot (q_i + q_f))^2}{16} \frac{1}{2m^2 M^2 ((p_f - p_i)^2)^2} \\ &\quad \left\{ p_i \cdot q_i p_f \cdot q_f + p_i \cdot q_f p_f \cdot q_i - m^2 q_i \cdot q_f - M^2 p_i \cdot p_f + 2m^2 M^2 \right\}. \end{aligned} \quad (23)$$

To further extract the physics of the two-particle scattering process we choose as coordinate system the one in which the particle with mass M is at rest, i.e.

$$\begin{aligned} q_i &= (M, 0), & q_f &= (M + E - E', \underline{p} - \underline{p}') \\ p_i &= (E, \underline{p}), & p_f &= (E', \underline{p}'). \end{aligned} \quad (24)$$

Energy conservation for the chosen coordinates relates the energy E' of the outgoing particle with mass m to the energy E of the incoming particle with mass m and to the scattering angle ϑ

$$E' = \frac{\frac{E}{M} + \left(\frac{m}{M}\right)^2 + \frac{|\underline{p}| |\underline{p}'|}{M^2} \cos \vartheta}{1 + \frac{E}{M}} M. \quad (25)$$

Performing the phase space integrals over q_f and E' in Eqn.(20) in the usual way [11] leaves us with the scattering cross-section in the rest mass frame of the particle with mass M

$$\frac{d\bar{\sigma}}{d\Omega'} = \frac{1}{(2\pi)^2} \frac{|\underline{p}'|}{|\underline{p}|} \frac{m^2 M}{M + E - \frac{|\underline{p}|}{|\underline{p}'|} E' \cos \vartheta} \overline{|\mathcal{M}_{fi}|^2} \quad (26)$$

which after a little algebra can be expressed in terms of E and E' as

$$\begin{aligned} \frac{d\bar{\sigma}}{d\Omega'} &= \frac{(E'^2 - m^2)^{3/2}}{(E^2 - m^2)^{1/2}} \frac{m^2}{E E' - m^2 (1 + \frac{E}{M} - \frac{E'}{M})} \\ &\quad \frac{(gL_P)^4}{(4\pi)^2} \frac{M^2 (E + E')^2}{4} \\ &\quad \frac{1}{2m^2 (E - E')^2} \left\{ \frac{E^2}{M^2} + \frac{E'^2}{M^2} - \left(1 + \frac{m^2}{M^2}\right) \left(\frac{E}{M} - \frac{E'}{M}\right) \right\}. \end{aligned} \quad (27)$$

The first and last lines above are exactly the same as in the case of scattering of two Dirac particles with different masses in quantum electrodynamics [11] whereas the middle line represents the energy-dependent gravitational interaction strength replacing the square of the fine structure constant $\alpha = \frac{e e'}{4\pi}$.

We next evaluate both the limits of a heavy scatterer $\frac{E}{M} \ll 1$ and of an ultra-relativistic incoming particle $\frac{m}{E} \ll 1$.

If the energy E of the incoming particle of mass m is much smaller than the mass M of the scatterer Eqn.(25) yields up to higher orders in $\frac{E}{M}$

$$\frac{E}{M} \ll 1 \Rightarrow E' = E, |\underline{p}'| = |\underline{p}|. \quad (28)$$

In addition we have from Eqn.(25) in this limit

$$\begin{aligned} E - E' &= 2 \frac{|\underline{p}|^2}{M} \sin^2 \frac{\vartheta}{2} \\ \{\dots\} &= 2 \frac{E^2}{M^2} - 2 \frac{|\underline{p}|^2}{M^2} \sin^2 \frac{\vartheta}{2}, \end{aligned} \quad (29)$$

where $\{\dots\}$ denotes the bracket appearing in Eqn.(27).

Setting

$$\beta = \frac{|\underline{p}|}{E} = |\underline{v}| \quad (30)$$

we find the analogue to the Mott scattering cross-section [11] in quantum gravity expressed in terms of β and ϑ

$$\frac{d\bar{\sigma}}{d\Omega'} = \frac{g^4}{(4\pi)^2} \Gamma^2 M^2 \frac{1 - \beta^2 \sin^2 \frac{\vartheta}{2}}{4 \beta^4 \sin^4 \frac{\vartheta}{2}} \quad (31)$$

recalling that $L_P = \sqrt{\Gamma}$.

Eqn.(31) reduces in the non-relativistic limit $\beta \rightarrow 0$ to the Rutherford-like formula

$$\frac{d\bar{\sigma}}{d\Omega'} = \frac{\Gamma^2 M^2}{4 |\underline{v}|^4 \sin^4 \frac{\vartheta}{2}} \quad (32)$$

obtained from a quantum mechanical (and incidentally a classical) treatment of the scattering of a particle of mass m off an infinitely heavy scatterer M in Newtonian gravity [12] if we fix the coupling constant to be

$$g^2 = 4\pi. \quad (33)$$

Note that the value of g^2 depends on the conventions chosen and that it is the dimensionless combination $(gL_P)^2 \frac{M(E+E')}{2} \ll 1$ which really matters and allows for a perturbative approach.

We again stress the fact that the scattering of two Dirac particles with different masses in the limit of a heavy scatterer and a non-relativistic incoming particle is physically equivalent to gravitational Rutherford scattering - and hence provides a non-trivial comparison and test for our claim that the theory presented in [1, 3, 10] is indeed a theory describing gravity at the quantum level (allowing us in the process to fix the numerical value of g^2 as well).

We finally note that Eqn.(31) does not depend on the specific properties of the incoming particle, but just on the kinematical factor β - an expression that the principle of equivalence holds in the above limit.

If on the other hand the energy E is much larger than the mass m of the incoming particle we have

$$\frac{m}{E} \ll 1 \Rightarrow E = |\underline{p}|, E' = |\underline{p}'| \quad (34)$$

and Eqn.(25) yields

$$\begin{aligned} E - E' &= 2 \frac{E E'}{M} \sin^2 \frac{\vartheta}{2} \\ \frac{E'}{E} &= \frac{1}{1 + 2 \frac{E}{M} \sin^2 \frac{\vartheta}{2}} \\ \{\dots\} &= \frac{2 E E'}{M^2} \left(\cos^2 \frac{\vartheta}{2} + \frac{2 \frac{E^2}{M^2} \sin^4 \frac{\vartheta}{2}}{1 + 2 \frac{E}{M} \sin^2 \frac{\vartheta}{2}} \right), \end{aligned} \quad (35)$$

where $\{\dots\}$ again denotes the bracket appearing in Eqn.(27).

A little algebra yields the scattering cross-section in this limit in terms of E and ϑ as

$$\begin{aligned} \frac{d\bar{\sigma}}{d\Omega'} &= \frac{(gL_P)^4}{(4\pi)^2} M^2 \frac{1}{4 \sin^4 \frac{\vartheta}{2}} \frac{\left(1 + \frac{E}{M} \sin^2 \frac{\vartheta}{2}\right)^2}{\left(1 + 2 \frac{E}{M} \sin^2 \frac{\vartheta}{2}\right)^3} \\ &\quad \left(\cos^2 \frac{\vartheta}{2} + \frac{2 \frac{E^2}{M^2} \sin^4 \frac{\vartheta}{2}}{1 + 2 \frac{E}{M} \sin^2 \frac{\vartheta}{2}} \right). \end{aligned} \quad (36)$$

Note that for a heavy scatterer $\frac{E}{M} \ll 1$ it reduces to Eqn.(31) in the relativistic limit $\beta \rightarrow 1$ as it should.

We finally note that Eqn.(36) does depend on the specific properties of the incoming particle, i.e. its mass m , as does the general formula Eqn.(27) for the scattering cross-section - an expression that the principle of equivalence seems not to generally hold in a quantum context.

4 Conclusions

In this paper within the gauge field theory of volume-preserving diffeomorphisms coupled to Dirac matter we have calculated to leading order in perturbation theory both the scattering amplitude and then the resulting scattering cross-section for two Dirac particles with different masses and inertial equal to gravitational energy-momenta.

We then have evaluated relevant limits for the resulting cross-section in the rest frame of one of the initial particles. In the non-relativistic limit for the second initial particle we have recovered the cross-section of a non-relativistic particle scattering off an infinitely heavy scatterer calculated quantum mechanically in Newtonian gravity.

Besides some general features of the present theory such as the universal coupling of the gauge field to any other fields this is a further indication that the gauge field theory of volume-preserving diffeomorphisms can be viewed as a renormalizable quantum theory of gravity.

A Notations and Conventions

Generally, (\mathbf{M}^4, η) denotes the four-dimensional Minkowski space with metric $\eta = \text{diag}(1, -1, -1, -1)$, small letters denote spacetime coordinates and parameters and capital letters denote coordinates and parameters in inner space.

Specifically, $x^\lambda, y^\mu, z^\nu, \dots$ denote Cartesian spacetime coordinates. The small Greek indices λ, μ, ν, \dots from the middle of the Greek alphabet run over $0, 1, 2, 3$. They are raised and lowered with η , i.e. $x_\mu = \eta_{\mu\nu} x^\nu$ etc. and transform covariantly w.r.t. the Lorentz group $SO(1, 3)$. Partial differentiation w.r.t to x^μ is denoted by $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$.

Working in Minkowskian gauges [3] $X^\alpha, Y^\beta, Z^\gamma, \dots$ denote inner Cartesian coordinates. The small Greek indices $\alpha, \beta, \gamma, \dots$ from the beginning of the Greek alphabet run again over $0, 1, 2, 3$. They are raised and lowered with the inner Minkowski metric η , i.e. $X_\alpha = \eta_{\alpha\beta} X^\beta$ etc. and transform covariantly w.r.t. the inner Lorentz group $SO(1, 3)$. Partial differentiation w.r.t to X^α is denoted by $\nabla_\alpha \equiv \frac{\partial}{\partial X^\alpha}$.

The same lower and upper indices are summed unless indicated otherwise.

All further conventions related e.g. to spinors, phase space integrals etc. are standard and taken from [7, 11].

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